

WHEN THE NUMBER OF DIVISORS IS A QUADRATIC RESIDUE

OLIVIER BORDELLÈS

ABSTRACT. Let $q > 2$ be a prime number and define $\lambda_q := \left(\frac{\tau}{q}\right)$ where $\tau(n)$ is the number of divisors of n and $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol. When $\tau(n)$ is a quadratic residue modulo q , then $(\lambda_q \star \mathbf{1})(n)$ could be close to the number of divisors of n . This is the aim of this work to compare the mean value of the function $\lambda_q \star \mathbf{1}$ to the well known average order of τ . The proof reveals that the results depend heavily on the value of $\left(\frac{2}{q}\right)$. A bound for short sums in the case $q = 5$ is also given, using profound results from the theory of integer points close to certain smooth curves.

1. INTRODUCTION AND MAIN RESULT

If $\lambda = (-1)^\Omega$ is the Liouville function, then

$$L(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)} \quad (\sigma > 1).$$

This implies the convolution identity

$$\sum_{n \leq x} (\lambda \star \mathbf{1})(n) = \left\lfloor x^{1/2} \right\rfloor.$$

Define $\lambda_3 := \left(\frac{\tau}{3}\right)$ where $\tau(n)$ is the number of divisors of n and $\left(\frac{\cdot}{3}\right)$ is the Legendre symbol modulo 3. Then from Proposition 3 below

$$L(s, \lambda_3) = \frac{\zeta(3s)}{\zeta(s)} \quad (\sigma > 1)$$

implying the convolution identity

$$\sum_{n \leq x} (\lambda_3 \star \mathbf{1})(n) = \left\lfloor x^{1/3} \right\rfloor.$$

Now let $q > 2$ be a prime number and define $\lambda_q := \left(\frac{\tau}{q}\right)$ where $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol modulo q . Our main aim is to investigate the sum

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n).$$

When $\tau(n)$ is a quadratic residue modulo q , one may wonder if $(\lambda_q \star \mathbf{1})(n)$ has a high probability to be equal to the number of divisors of n . It then could be interesting to study its average order and to compare it to that of τ , i.e.

$$(1) \quad \sum_{n \leq x} \tau(n) = x(\log x + 2\gamma - 1) + O(x^{\theta+\varepsilon})$$

where $\frac{1}{4} \leq \theta \leq \frac{131}{416}$, the left-hand side being established by Hardy [5], the right-hand side being the best estimate to date due to Huxley [6]. The main result of this paper can be stated as follows.

Theorem 1. *Let $q > 3$ be a prime number.*

▷ *If $q \equiv \pm 1 \pmod{8}$*

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) = x\zeta(q)P_q(1) \left\{ \log x + 2\gamma - 1 + q \frac{\zeta'(q)}{\zeta(q)} + \frac{P'_q(1)}{P_q(1)} \right\} + O_{q,\varepsilon} \left(x^{\max(1/c_q, \theta) + \varepsilon} \right)$$

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where θ is defined in (1), c_q is given in (2) and

$$P_q(1) = \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{1}{p^m} \right)$$

$$\frac{P'_q}{P_q}(1) = - \sum_p \log p \left(\frac{\sum_{m=c_q}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{m}{p^m}}{1 + \sum_{m=c_q}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{1}{p^m}} \right).$$

▷ If $q \equiv \pm 11 \pmod{24}$

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) = x^{1/2} \zeta\left(\frac{q}{2}\right) R_q\left(\frac{1}{2}\right) + O_{q,\varepsilon}\left(x^{1/3+\varepsilon}\right)$$

where

$$R_q\left(\frac{1}{2}\right) := \prod_p \left(1 + \sum_{m=3}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{m/2}} \right).$$

▷ If $q \equiv \pm 5 \pmod{24}$, there exists $c > 0$ such that

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll_q x^{1/2} e^{-c(\log x^{1/4})^{3/5}} (\log \log x^{1/4})^{-1/5}.$$

Furthermore, if the Riemann hypothesis is true, then for x sufficiently large

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll_{q,\varepsilon} x^{1/4} e^{(\log \sqrt{x})^{1/2}} (\log \log \sqrt{x})^{5/2+\varepsilon}.$$

Example 2.

$$\sum_{n \leq x} (\lambda_7 \star \mathbf{1})(n) \doteq 0.454 x (\log x + 2\gamma + 0.784) + O_\varepsilon\left(x^{1/2+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_{23} \star \mathbf{1})(n) \doteq 0.899 x (\log x + 2\gamma - 0.678) + O_\varepsilon\left(x^{131/416+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_{13} \star \mathbf{1})(n) \doteq 1.969 x^{1/2} + O_\varepsilon\left(x^{1/3+\varepsilon}\right).$$

$$\sum_{n \leq x} (\lambda_5 \star \mathbf{1})(n) \ll x^{1/2} e^{-c(\log x^{1/4})^{3/5}} (\log \log x^{1/4})^{-1/5}.$$

2. NOTATION

In what follows, $x \geq e^4$ is a large real number, $\varepsilon \in (0, \frac{1}{4})$ is a small real number which does not need to be the same at each occurrence, $s := \sigma + it \in \mathbb{C}$, q always denotes an odd prime number, $\left(\frac{\cdot}{q}\right)$ is the Legendre symbol modulo q and define

$$\lambda_q := \left(\frac{\tau}{q}\right)$$

where $\tau(n) := \sum_{d|n} 1$. Also, $\mathbf{1}$ is the constant arithmetic function equal to 1.

For any arithmetic functions F and G , $L(s, F)$ is the Dirichlet series of F , the Dirichlet convolution product $F \star G$ is defined by

$$(F \star G)(n) := \sum_{d|n} F(d)G(n/d)$$

and F^{-1} is the Dirichlet convolution inverse of F . If $r \in \mathbb{Z}_{\geq 2}$, then

$$a_r(n) := \begin{cases} 1, & \text{if } n = m^r; \\ 0, & \text{otherwise.} \end{cases}$$

For some $c > 0$, set

$$\delta_c(x) := e^{-c(\log x)^{3/5}} (\log \log x)^{-1/5} \quad \text{and} \quad \omega(x) := e^{(\log x)^{1/2}} (\log \log x)^{5/2+\varepsilon}.$$

Finally, let $M(x)$ and $L(x)$ be respectively the Mertens function and the summatory function of the Liouville function, i.e.

$$M(x) := \sum_{n \leq x} \mu(n) \quad \text{and} \quad L(x) := \sum_{n \leq x} \lambda(n).$$

3. THE DIRICHLET SERIES OF λ_q

Proposition 3. *Let $q \geq 3$ be a prime number. For any $s \in \mathbb{C}$ such that $\sigma > 1$*

▷ *If $q \equiv \pm 1 \pmod{8}$*

$$L(s, \lambda_q) = \zeta(qs)\zeta(s) \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right)$$

where

$$(2) \quad c_q := \begin{cases} 2, & \text{if } q \equiv \pm 7 \pmod{24}; \\ \geq 4, & \text{if } q \equiv \pm 1 \pmod{24}. \end{cases}$$

▷ *If $q \equiv \pm 3 \pmod{8}$*

$$L(s, \lambda_q) = \frac{\zeta(qs)\zeta(2s)}{\zeta(s)} \prod_p \left(1 + \sum_{m=d_q}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right)$$

where

$$d_q := \begin{cases} 2, & \text{if } q \equiv \pm 5 \pmod{24} \text{ or } q = 3; \\ 3, & \text{if } q \equiv \pm 11 \pmod{24}. \end{cases}$$

Proof. Set $\chi_q := \left(\frac{\cdot}{q} \right)$ for convenience. From [8, Lemma 2.1], we have

$$\begin{aligned} L(s, \lambda_q) &= \prod_p \left(1 + \sum_{\alpha=1}^{\infty} \frac{\chi_q(\alpha+1)}{p^{s\alpha}} \right) = \prod_p \left(1 + p^s \sum_{\alpha=2}^{\infty} \frac{\chi_q(\alpha)}{p^{s\alpha}} \right) \\ &= \prod_p \left\{ 1 + p^s \left(\left(1 - \frac{1}{p^{qs}} \right)^{-1} \sum_{m=1}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{ms}} - p^{-s} \right) \right\} \\ &= \prod_p \left\{ \left(1 - \frac{1}{p^{qs}} \right)^{-1} \sum_{m=1}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right\} \\ &= \zeta(qs) \prod_p \left(1 + \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right). \end{aligned}$$

If $q \equiv \pm 1 \pmod{8}$, then $\left(\frac{2}{q} \right) = 1$ and

$$L(s, \lambda_q) = \zeta(qs)\zeta(s) \prod_p \left(1 - \frac{1}{p^s} + \left(1 - \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right)$$

where

$$\begin{aligned} \left(1 - \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{(m-1)s}} &= \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \left(\frac{1}{p^{(m-1)s}} - \frac{1}{p^{ms}} \right) \\ &= \sum_{m=1}^{q-2} \left(\frac{m+1}{q} \right) \frac{1}{p^{ms}} - \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{ms}} \\ &= \left(\frac{2}{q} \right) \frac{1}{p^s} + \sum_{m=2}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} - \left(\frac{q}{q} \right) \frac{1}{p^{(q-1)s}} \\ &= \sum_{m=2}^{q-1} \left\{ \left(\frac{m+1}{q} \right) - \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} + \frac{1}{p^s}. \end{aligned}$$

Similarly, if $q \equiv \pm 3 \pmod{8}$, then $\left(\frac{2}{q} \right) = -1$ and

$$L(s, \lambda_q) = \frac{\zeta(qs)\zeta(2s)}{\zeta(s)} \prod_p \left(1 + \frac{1}{p^s} + \left(1 + \frac{1}{p^s} \right) \sum_{m=2}^{q-1} \left(\frac{m}{q} \right) \frac{1}{p^{(m-1)s}} \right)$$

where

$$\begin{aligned}
\left(1 + \frac{1}{p^s}\right) \sum_{m=2}^{q-1} \left(\frac{m}{q}\right) \frac{1}{p^{(m-1)s}} &= \sum_{m=2}^{q-1} \left(\frac{m}{q}\right) \left(\frac{1}{p^{(m-1)s}} + \frac{1}{p^{ms}}\right) \\
&= \sum_{m=1}^{q-2} \left(\frac{m+1}{q}\right) \frac{1}{p^{ms}} + \sum_{m=2}^{q-1} \left(\frac{m}{q}\right) \frac{1}{p^{ms}} \\
&= \left(\frac{2}{q}\right) \frac{1}{p^s} + \sum_{m=2}^{q-1} \left\{ \left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} - \left(\frac{q}{q}\right) \frac{1}{p^{(q-1)s}} \\
&= \sum_{m=2}^{q-1} \left\{ \left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} - \frac{1}{p^s}.
\end{aligned}$$

We achieve the proof noting that, if $q \equiv \pm 1 \pmod{24}$, then $\left(\frac{3}{q}\right) - \left(\frac{2}{q}\right) = \left(\frac{4}{q}\right) - \left(\frac{3}{q}\right) = 0$ and, similarly, if $q \equiv \pm 11 \pmod{24}$, then $\left(\frac{3}{q}\right) + \left(\frac{2}{q}\right) = 0$ whereas $\left(\frac{4}{q}\right) + \left(\frac{3}{q}\right) = 2$. \square

4. PROOF OF THEOREM 1

4.1. **The case** $q \equiv \pm 1 \pmod{8}$. For $\sigma > 1$, we set

$$G_q(s) = \zeta(qs) \prod_p \left(1 + \sum_{m=c_q}^{q-1} \left\{ \left(\frac{m+1}{q}\right) - \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} \right) := \zeta(qs) P_q(s) := \sum_{n=1}^{\infty} \frac{g_q(n)}{n^s}.$$

First observe that $c_q < q$ in the case $q \equiv \pm 1 \pmod{24}$. Indeed, among the $q-4$ integers $m \in \{4, \dots, q-1\}$, it is known from [3, p.76] that there are $\frac{1}{2}(q-3)-3$ of them such that $\left(\frac{m}{q}\right) = \left(\frac{m+1}{q}\right)$. Consequently there are $\frac{1}{2}(q+1)$ integers $m \in \{4, \dots, q-1\}$ verifying $\left(\frac{m}{q}\right) \neq \left(\frac{m+1}{q}\right)$, and the inequality follows.

Thus this Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{1}{c_q}$ where c_q is given in (2), so that

$$\sum_{n \leq x} |g_q(n)| \ll_{q,\varepsilon} x^{1/c_q + \varepsilon}.$$

By partial summation, we infer

$$\begin{aligned}
\sum_{n \leq x} \frac{g_q(n)}{n} &= \zeta(q) P_q(1) + O\left(x^{-1+1/c_q + \varepsilon}\right) \\
\sum_{n \leq x} \frac{g_q(n)}{n} \log \frac{x}{n} &= \zeta(q) P_q(1) \log x + q P_q(1) \zeta'(q) + P_q'(1) \zeta(q) + O\left(x^{-1+1/c_q + \varepsilon}\right).
\end{aligned}$$

From Proposition 3, $\lambda_q \star \mathbf{1} = g_q \star \tau$. Consequently

$$\begin{aligned}
\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) &= \sum_{d \leq x} g_q(d) \sum_{k \leq x/d} \tau(k) \\
&= \sum_{d \leq x} g_q(d) \left\{ \frac{x}{d} \log \frac{x}{d} + (2\gamma - 1) \frac{x}{d} + O\left(\left(\frac{x}{d}\right)^{\theta + \varepsilon}\right) \right\} \\
&= x \left\{ \zeta(q) P_q(1) \log x + q P_q(1) \zeta'(q) + P_q'(1) \zeta(q) + (2\gamma - 1) \zeta(q) P_q(1) \right\} \\
&\quad + O\left(x^{\max(1/c_q, \theta) + \varepsilon}\right)
\end{aligned}$$

where θ is defined in (1) and where we used

$$x^{-\varepsilon} \sum_{d \leq x} \frac{|g_q(d)|}{d^\theta} \ll \begin{cases} x^{1/c_q - \theta}, & \text{if } c_q^{-1} \geq \theta; \\ 1, & \text{otherwise.} \end{cases}$$

4.2. **The case** $q \equiv \pm 11 \pmod{24}$. For $\sigma > 1$, we set

$$H_q(s) = \zeta(qs) \prod_p \left(1 + \sum_{m=3}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right) := \zeta(qs) R_q(s) := \sum_{n=1}^{\infty} \frac{h_q(n)}{n^s}.$$

Since $q > 5$, this Dirichlet series is absolutely convergent in the half-plane $\sigma > \frac{1}{3}$, so that

$$\sum_{n \leq x} |h_q(n)| \ll_{q,\varepsilon} x^{1/3+\varepsilon}.$$

From Proposition 3, $\lambda_q \star \mathbf{1} = h_q \star a_2$, hence

$$\begin{aligned} \sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) &= \sum_{d \leq x} h_q(d) \left\lfloor \sqrt{\frac{x}{d}} \right\rfloor \\ &= x^{1/2} \sum_{d \leq x} \frac{h_q(d)}{\sqrt{d}} + O(x^{1/3+\varepsilon}) \\ &= x^{1/2} H_q\left(\frac{1}{2}\right) + O(x^{1/3+\varepsilon}). \end{aligned}$$

4.3. **The case** $q \equiv \pm 5 \pmod{24}$. In this case, it is necessary to rewrite $L(s, \lambda_q)$ in the following shape.

Lemma 4. Assume $q \equiv \pm 5 \pmod{24}$. For any $\sigma > 1$, $L(s, \lambda_q) = \frac{K_q(s)}{\zeta(s)\zeta(2s)}$ with

$$K_q(s) := \begin{cases} \zeta(5s), & \text{if } q = 5 \\ \zeta(4s)L_q(s), & \text{if } q \equiv \pm 19, \pm 29 \pmod{120} \\ \frac{\mathcal{L}_q(s)}{\zeta(4s)}, & \text{if } q \equiv \pm 43, \pm 53 \pmod{120} \end{cases}$$

where

$$L_q(s) := \zeta(qs) \prod_p \left(1 + \frac{2(p^{2s} + p^s + 1)}{p^{7s} - p^{5s}} + \frac{p^{2s} + 1}{p^{2s} - 1} \sum_{m=6}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right)$$

and

$$\begin{aligned} \mathcal{L}_q(s) &:= \zeta(qs) \prod_p \left(1 - \frac{2p^{2s} - 1}{(p^{2s} - 1)^3 (p^{2s} + 1)} \right. \\ &\quad \left. + \frac{p^{8s}}{(p^{2s} - 1)^3 (p^{2s} + 1)} \sum_{m=6}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right). \end{aligned}$$

The Dirichlet series L_q is absolutely convergent in the half-plane $\sigma > \frac{1}{5}$, and the Dirichlet series \mathcal{L}_q is absolutely convergent in the half-plane $\sigma > \frac{1}{6}$.

Proof. From Proposition 3, we immediately get

$$(3) \quad L(s, \lambda_5) = \frac{\zeta(5s)}{\zeta(s)\zeta(2s)}.$$

Now suppose $q > 5$ and $q \equiv \pm 5 \pmod{24}$. In this case, $\left(\frac{3}{q}\right) + \left(\frac{2}{q}\right) = -2$ and $\left(\frac{4}{q}\right) + \left(\frac{3}{q}\right) = 0$ so that we may write by Proposition 3

$$\begin{aligned} L(s, \lambda_q) &= \frac{\zeta(qs)\zeta(2s)}{\zeta(s)} \prod_p \left(1 - \frac{2}{p^{2s}} + \sum_{m=4}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right) \\ &= \frac{K_q(s)}{\zeta(s)\zeta(2s)} \end{aligned}$$

where

$$K_q(s) := \zeta(qs) \prod_p \left(1 - \frac{1}{(p^{2s} - 1)^2} + \frac{p^{4s}}{(p^{2s} - 1)^2} \sum_{m=4}^{q-1} \left\{ \left(\frac{m+1}{q} \right) + \left(\frac{m}{q} \right) \right\} \frac{1}{p^{ms}} \right).$$

Assume $q \equiv \pm 19, \pm 29 \pmod{120}$. Then

$$\left(\frac{5}{q}\right) + \left(\frac{4}{q}\right) = \left(\frac{6}{q}\right) + \left(\frac{5}{q}\right) = 2.$$

$K_q(s)$ can therefore be written as

$$\begin{aligned} K_q(s) &= \zeta(qs) \prod_p \left(1 + \frac{p^s + 2}{p^s(p^{2s} - 1)^2} + \frac{p^{4s}}{(p^{2s} - 1)^2} \sum_{m=6}^{q-1} \left\{ \left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} \right) \\ &= \zeta(qs) \zeta(4s) \prod_p \left(1 + \frac{2(p^{2s} + p^s + 1)}{p^{7s} - p^{5s}} + \frac{p^{2s} + 1}{p^{2s} - 1} \sum_{m=6}^{q-1} \left\{ \left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} \right) \\ &= \zeta(4s) L_q(s). \end{aligned}$$

Similarly, if $q \equiv \pm 43, \pm 53 \pmod{120}$, then

$$\left(\frac{5}{q}\right) + \left(\frac{4}{q}\right) = \left(\frac{6}{q}\right) + \left(\frac{5}{q}\right) = 0.$$

Hence

$$\begin{aligned} K_q(s) &:= \zeta(qs) \prod_p \left(1 - \frac{1}{(p^{2s} - 1)^2} + \frac{p^{4s}}{(p^{2s} - 1)^2} \sum_{m=6}^{q-1} \left\{ \left(\frac{m+1}{q}\right) + \left(\frac{m}{q}\right) \right\} \frac{1}{p^{ms}} \right) \\ &= \frac{\mathcal{L}_q(s)}{\zeta(4s)}. \end{aligned}$$

The proof is complete. \square

We now are in a position to prove Theorem 1 in the case $q \equiv \pm 5 \pmod{24}$.

Assume first that $q \equiv \pm 19, \pm 29 \pmod{120}$ and let $\ell_q(n)$ be the n -th coefficient of the Dirichlet series $L_q(s)$. From Lemma 4, $\lambda_q \star \mathbf{1} = \ell_q \star a_4 \star a_2^{-1}$ and therefore

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) = \sum_{d \leq x} \ell_q(d) \sum_{m \leq (x/d)^{1/4}} M\left(\frac{1}{m^2} \sqrt{\frac{x}{d}}\right) = \sum_{d \leq x} \ell_q(d) L\left(\sqrt{\frac{x}{d}}\right).$$

Since $L(z) \ll z \delta_c(z)$ for some $c > 0$

$$\begin{aligned} \sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) &\ll x^{1/2} \sum_{d \leq x} \frac{|\ell_q(d)|}{\sqrt{d}} \delta_c\left(\sqrt{\frac{x}{d}}\right) \\ &\ll x^{1/2} \left(\sum_{d \leq \sqrt{x}} + \sum_{\sqrt{x} < d \leq x} \right) \frac{|\ell_q(d)|}{\sqrt{d}} \delta_c\left(\sqrt{\frac{x}{d}}\right) \\ &\ll x^{1/2} \delta_c(x^{1/4}) + x^{1/2} \sum_{d > \sqrt{x}} \frac{|\ell_q(d)|}{\sqrt{d}}. \end{aligned}$$

The Dirichlet series $L_q(s) := \sum_{n=1}^{\infty} \ell_q(n) n^{-s}$ is absolutely convergent in the half-plane $\sigma > \frac{1}{5}$, consequently

$$\sum_{d \leq z} |\ell_q(d)| \ll_{q,\varepsilon} z^{1/5+\varepsilon}$$

and by partial summation

$$\sum_{d > z} \frac{|\ell_q(d)|}{\sqrt{d}} \ll_{q,\varepsilon} z^{-3/10+\varepsilon}.$$

We infer that

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/2} \delta_c(x^{1/4}) + x^{7/20+\varepsilon} \ll x^{1/2} \delta_c(x^{1/4}).$$

Now suppose that the Riemann hypothesis is true. By [1], which is a refinement of [9], we know that $M(z) \ll_{\varepsilon} z^{1/2} \omega(z)$. The method of [9, 1] may be adapted to the function L yielding

$$L(z) \ll_{\varepsilon} z^{1/2} \omega(z) \log z.$$

Observe that, for any $a \geq 2$, $\varepsilon > 0$ and $z \geq e^{e^{\varepsilon}}$

$$\log z \exp\left(\sqrt{\log z} (\log \log z)^a\right) \leq \exp\left(\sqrt{\log z} (\log \log z)^{a+\varepsilon}\right)$$

so that $L(z) \ll_{\varepsilon} z^{1/2} \omega(z)$ and hence

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/4} \sum_{d \leq x} \frac{|\ell_q(d)|}{d^{1/4}} \omega\left(\sqrt{\frac{x}{d}}\right) \ll x^{1/4} \omega(\sqrt{x}) \sum_{d \leq x} \frac{|\ell_q(d)|}{d^{1/4}} \ll x^{1/4} \omega(\sqrt{x})$$

achieving the proof in that case. The case $q = 5$ is similar but simpler since $\lambda_5 \star \mathbf{1} = a_5 \star a_2^{-1}$ by (3).

Finally, when $q \equiv \pm 43, \pm 53 \pmod{120}$, we proceed as above. Let $\nu_q(n)$ be the n -th coefficient of the Dirichlet series $\mathcal{L}_q(s)$. Then $\lambda_q \star \mathbf{1} = \nu_q \star a_4^{-1} \star a_2^{-1}$ from Lemma 4, so that

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) = \sum_{d \leq x} \nu_q(d) \sum_{m \leq (x/d)^{1/4}} \mu(m) M\left(\frac{1}{m^2} \sqrt{\frac{x}{d}}\right)$$

and estimating trivially yields

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/2} \sum_{d \leq x} \frac{|\nu_q(d)|}{\sqrt{d}} \sum_{m \leq (x/d)^{1/4}} \frac{1}{m^2} \delta_c\left(\frac{1}{m^2} \sqrt{\frac{x}{d}}\right)$$

and we complete the proof as in the previous case. \square

Remark 5. Let us stress that a bound of the shape

$$\sum_{n \leq x} (\lambda_q \star \mathbf{1})(n) \ll x^{1/4+\varepsilon}$$

for all x sufficiently large and small $\varepsilon > 0$, is a necessary and sufficient condition for the Riemann hypothesis. Indeed, if this estimate holds, then by partial summation the series $\sum_{n=1}^{\infty} (\lambda_q \star \mathbf{1})(n) n^{-s}$ is absolutely convergent in the half-plane $\sigma > \frac{1}{4}$. Consequently, the function $K_q(s) \zeta(2s)^{-1}$ is analytic in this half-plane. In particular, $\zeta(2s)$ does not vanish in this half-plane, implying the Riemann hypothesis, proving the necessary condition, the sufficiency being established above.

5. A SHORT INTERVAL RESULT FOR THE CASE $q = 5$

5.1. Introduction. This section deals with sums of the shape

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n)$$

where $x^{\varepsilon} \leq y \leq x$. From Theorem 1

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll x^{1/2} e^{-c(\log x^{1/4})^{3/5}} (\log \log x^{1/4})^{-1/5}$$

and if the Riemann hypothesis is true, then

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll_{\varepsilon} x^{1/4} e^{(\log \sqrt{x})^{1/2} (\log \log \sqrt{x})^{5/2+\varepsilon}}.$$

The purpose is to improve significantly upon these estimates when $y = o(x)$, by using fine results belonging to the theory of integer points near a suitably chosen smooth curve. To this end, we need the following additional specific notation. Let $\delta \in (0, \frac{1}{4})$, $N \in \mathbb{Z}_{\geq 1}$ large, $f : [N, 2N] \rightarrow \mathbb{R}$ be any map, and define $\mathcal{R}(f, N, \delta)$ to be the number of elements of the set of integers $n \in [N, 2N]$ such that $\|f(n)\| < \delta$, where $\|x\|$ is the distance from x to its nearest integer. Note that the trivial bound is given by

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll \sum_{x < n \leq x+y} \tau(n) \ll y \log x.$$

5.2. Tools from the theory. In what follows, $N \in \mathbb{Z}_{\geq 1}$ is large and $\delta \in (0, \frac{1}{4})$. The first result is [7, Theorem 5] with $k = 5$. See also [2, Theorem 5.23 (iv)].

Lemma 6 (5th derivative test). *Let $f \in C^5[N, 2N]$ such that there exist $\lambda_4 > 0$ and $\lambda_5 > 0$ satisfying $\lambda_4 = N\lambda_5$ and, for any $x \in [N, 2N]$*

$$|f^{(4)}(x)| \asymp \lambda_4 \quad \text{and} \quad |f^{(5)}(x)| \asymp \lambda_5.$$

Then

$$\mathcal{R}(f, N, \delta) \ll N\lambda_5^{1/15} + N\delta^{1/6} + (\delta\lambda_4^{-1})^{1/4} + 1.$$

Remark 7. The basic result of the theory is the following first derivative test (see [2, Theorem 5.6]): Let $f \in C^1[N, 2N]$ such that there exist $\lambda_1 > 0$ such that $|f'(x)| \asymp \lambda_1$. Then

$$(4) \quad \mathcal{R}(f, N, \delta) \ll N\lambda_1 + N\delta + \delta\lambda_1^{-1} + 1.$$

This result is essentially a consequence of the mean value theorem.

The second tool is [4, Theorem 7] with $k = 3$.

Lemma 8. Let $s \in \mathbb{Q}^* \setminus \{\pm 2, \pm 1\}$ and $X > 0$ such that $N \leq X^{1/s}$. Then there exists a constant $c_3 := c_3(s) \in (0, \frac{1}{4})$ depending only on s such that, if

$$(5) \quad N^2\delta \leq c_3$$

then

$$\mathcal{R}\left(\frac{X}{n^s}, N, \delta\right) \ll (XN^{3-s})^{1/7} + \delta(XN^{59-s})^{1/21}.$$

Our last result relies the short sum of $\lambda_5 \star \mathbf{1}$ to a problem of counting integer points near a smooth curve.

Lemma 9. Let $1 \leq y \leq x$. Then

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll \max_{(16y^2x^{-1})^{1/5} < N \leq (2x)^{1/5}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}}\right) \log x + yx^{-1/2} + x^{-1/5}y^{2/5}.$$

Proof. Using (3), we get

$$\sum_{n \leq x} (\lambda_5 \star \mathbf{1})(n) = \sum_{d \leq \sqrt{x}} \mu(d) \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor$$

so that

$$\begin{aligned} \sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) &= \sum_{d \leq \sqrt{x}} \mu(d) \left(\left\lfloor \left(\frac{x+y}{d^2}\right)^{1/5} \right\rfloor - \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor \right) + \sum_{\sqrt{x} < d \leq \sqrt{x+y}} \mu(d) \\ &\ll \sum_{d \leq \sqrt{x}} \left(\left\lfloor \left(\frac{x+y}{d^2}\right)^{1/5} \right\rfloor - \left\lfloor \left(\frac{x}{d^2}\right)^{1/5} \right\rfloor \right) + yx^{-1/2} \\ &\ll \sum_{d \leq \sqrt{x}} \sum_{x < d^2 n^5 \leq x+y} 1 + yx^{-1/2} \\ &\ll \sum_{n \leq (2x)^{1/5} \left(\frac{x}{n^5}\right)^{1/2} < d \leq \left(\frac{x+y}{n^5}\right)^{1/2}} 1 + yx^{-1/2} \\ &\ll \sum_{(16y^2x^{-1})^{1/5} < n \leq (2x)^{1/5}} \left(\left\lfloor \sqrt{\frac{x+y}{n^5}} \right\rfloor - \left\lfloor \sqrt{\frac{x}{n^5}} \right\rfloor \right) + x^{-1/5}y^{2/5} + yx^{-1/2} \end{aligned}$$

and for any integers $N \in \left] (16y^2x^{-1})^{1/5}, (2x)^{1/5} \right]$ and $n \in [N, 2N]$

$$\sqrt{\frac{x+y}{n^5}} - \sqrt{\frac{x}{n^5}} < \frac{y}{\sqrt{N^5x}} < \frac{1}{4}$$

so that the sum does not exceed

$$\ll \max_{(16y^2x^{-1})^{1/5} < N \leq (2x)^{1/5}} \mathcal{R}\left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}}\right) \log x + x^{-1/5}y^{2/5} + yx^{-1/2}$$

as asserted. \square

5.3. The main result.

Theorem 10. Assume $y \leq c_3 x^{11/20}$ where $c_3 := c_3(\frac{5}{2})$ is given in (5). Then

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll \left(x^{1/12} + yx^{-4/9}\right) \log x.$$

Furthermore, if $y \leq c_3 x^{19/36}$

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll x^{1/12} \log x.$$

Proof. We split the first term in Lemma 9 into three parts, according to the ranges

$$(16y^2x^{-1})^{1/5} < N \leq 2x^{1/10}, \quad 2x^{1/10} < N \leq 2x^{1/6} \quad \text{and} \quad 2x^{1/6} < N \leq (2x)^{1/5}.$$

In the first case, we use Lemma 6 with $\lambda_4 = (xN^{-13})^{1/2}$ and $\lambda_5 = (xN^{-15})^{1/2}$ which yields

$$\max_{(16y^2x^{-1})^{1/5} < N \leq 2x^{1/10}} \mathcal{R} \left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}} \right) \ll x^{1/12} + x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4}.$$

For the second range, we use Lemma 8 with $X = x^{1/2}$, $s = \frac{5}{2}$ and $\delta = y(N^5x)^{-1/2}$. Notice that the conditions $N > 2x^{1/10}$ and $y \leq c_3 x^{11/20}$ ensure that $\delta < \frac{1}{4}$ and $N^2\delta \leq c_3$. We get

$$\max_{2x^{1/10} < N \leq 2x^{1/6}} \mathcal{R} \left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}} \right) \ll x^{1/12} + yx^{-4/9}.$$

The last range is easily treated with (4), giving

$$\max_{2x^{1/6} < N \leq (2x)^{1/5}} \mathcal{R} \left(\sqrt{\frac{x}{n^5}}, N, \frac{y}{\sqrt{N^5x}} \right) \ll x^{1/12} + yx^{-3/4}.$$

Using Lemma 9, we finally get

$$\sum_{x < n \leq x+y} (\lambda_5 \star \mathbf{1})(n) \ll \left(x^{1/12} + x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4} + yx^{-4/9} \right) \log x + x^{-1/5}y^{2/5}$$

and note that $x^{-1/40}y^{1/6} + x^{-3/20}y^{1/4} + x^{-1/5}y^{2/5} \ll x^{1/12}$ as soon as $y \leq x^{13/20}$. This completes the proof of the first estimate, the second one being obvious. \square

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REFERENCES

- [1] M. Balazard and A. de Roton, Notes de lecture de l'article "Partial sums of the Möbius function" de Kannan Soundararajan. *arXiv.org*, 2008, arXiv:0810.3587v1.
- [2] O. Bordellès, *Arithmetic Tales*, Springer, 2012.
- [3] H. Davenport, *The Higher Arithmetic*, 5th edition, Cambridge University Press, London, New York, 1982.
- [4] M. Filaseta and O. Trifonov, The distribution of fractional parts with applications to gap results in number theory, *Proc. London Math. Soc.* **73**(3) (1996), 241–278.
- [5] G. H. Hardy, On Dirichlet's divisor problem, *Proc. London Math. Soc.* **15** (1916), 1–25.
- [6] M. N. Huxley, Exponential sums and lattice points III, *Proc. London Math. Soc.* **87** (2003), 591–609.
- [7] M. N. HUXLEY & P. SARGOS, Points entiers au voisinage d'une courbe plane de classe C^n , II, *Functiones et Approximatio* **35** (2006), 91–115.
- [8] R. K. Muthumalai, Note on Legendre symbols connecting with certain infinite series, *Notes on Number Theory and Discrete Mathematics* **19** (2013), 77–83.
- [9] K. Soundararajan, Partial sums of the Möbius function, *J. Reine Angew. Math.* **631** (2009), 141–152.

2 ALLÉE DE LA COMBE, 43000 AIGUILHE, FRANCE
E-mail address: borde43@wanadoo.fr